Problem 8) a) By definition, the integral of $f'(z)$ along a given path is the sum of terms such as $[f(z_n+\Delta z_n)-f(z_n)]/\Delta z_n$ multiplied by Δz_n, in the limit when all Δz_n approach zero. (It is being assumed here that the continuous path is broken into infinitesimal segments at successive points $z_1 = z_\text{a}, z_2, z_3, \cdots, z_n, \cdots, z_N = z_\text{b}$.) Upon adding up the above terms, all the intermediate $f(z_n)$ drop out, leaving behind $-f(z_\text{a})$ from the first term, and $f(z_\text{b})$ from the last term. The final result of integration, therefore, is $\int_{z_\text{a}}^{z_\text{b}} f'(z)\mathrm{d}z = f(z_\text{b}) - f(z_\text{a})$. If the path happens to be closed, then $z_\text{a} = z_\text{b}$ and, consequently, $\int_{z_\text{a}}^{z_\text{b}} f'(z)\mathrm{d}z = 0$.

b) Since $[f(z)g(z)]' = f'(z)g(z) + f(z)g'(z)$, integrating both sides of this equation yields

$$\int_{z_\text{a}}^{z_\text{b}} [f(z)g(z)]'\mathrm{d}z = f(z_b)g(z_b) - f(z_a)g(z_a) = \int_{z_\text{a}}^{z_\text{b}} f'(z)g(z)\mathrm{d}z + \int_{z_\text{a}}^{z_\text{b}} f(z)g'(z)\mathrm{d}z.$$

The above identity leads immediately to the standard formula for integration by parts, namely,

$$\int_{z_\text{a}}^{z_\text{b}} f(z)g'(z)\mathrm{d}z = [f(z_b)g(z_b) - f(z_a)g(z_a)] - \int_{z_\text{a}}^{z_\text{b}} f'(z)g(z)\mathrm{d}z.$$

In the special case of a closed loop, where $z_\text{a} = z_\text{b}$, we will have

$$\int_{z_\text{a}}^{z_\text{b}} f(z)g'(z)\mathrm{d}z = -\int_{z_\text{a}}^{z_\text{b}} f'(z)g(z)\mathrm{d}z.$$

c) The function $f'(z)g(z) = a\exp(az)/(z - z_0)$ is analytic everywhere except at the first-order pole $z = z_0$. The integral of $f'(z)g(z)$ around a closed loop is therefore zero, unless the loop happens to enclose the pole at $z = z_0$, in which case the loop integral (in the counterclockwise direction) will be $\mathrm{i}2\pi a\exp(az_0)$. In contrast, the function $f(z)g'(z) = -\exp(az)/(z - z_0)^2$ is analytic everywhere except at the second-order pole $z = z_0$, where the corresponding residue is $-a\exp(az_0)$. Once again, the loop integral is seen to be zero unless the loop encloses the pole at $z = z_0$, in which case the (counterclockwise) integral of $f(z)g'(z)$ is $-\mathrm{i}2\pi a\exp(az_0)$. This integral, of course, is equal in magnitude and opposite in sign to the loop integral of $f'(z)g(z)$.